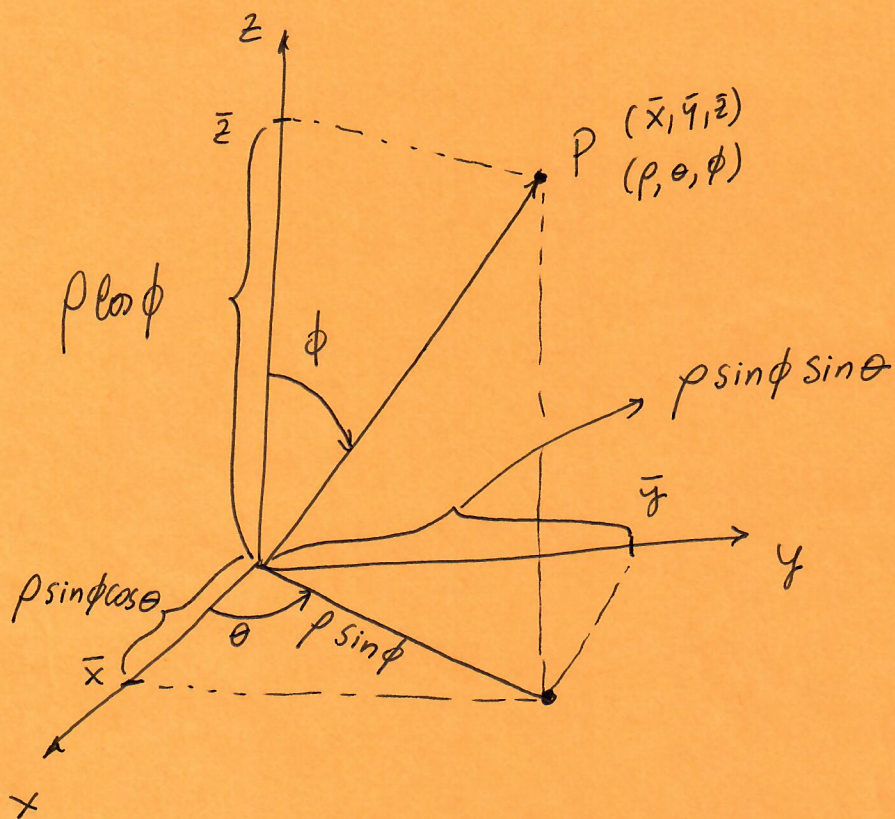


SPHERICAL COORDINATES.

$$\bar{x} = \rho \sin \phi \cos \theta$$

$$\bar{y} = \rho \sin \phi \sin \theta$$

$$\bar{z} = \rho \cos \phi$$

$$\rho \geq 0$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \pi$$

15.9

Spherical Coordinates.a) Description of Sphere:

$$x^2 + y^2 + z^2 = 1 \Rightarrow \rho^2 = 1 \Rightarrow \rho = 1$$

b) Description of half cone:

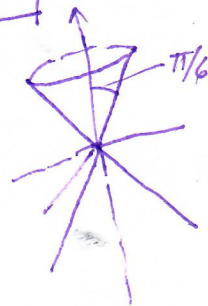
$$z = \sqrt{3x^2 + 3y^2} \Rightarrow \rho \cos \phi = \sqrt{3} \sqrt{\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta}$$

$$\text{or } \rho \cos \phi = \sqrt{3} \sqrt{\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta)} = \sqrt{3} \sqrt{\rho^2 \sin^2 \phi} = \sqrt{3} \rho \sin \phi$$

$$\text{or Cone: } \rho \cos \phi = \sqrt{3} \sin \phi \rho$$

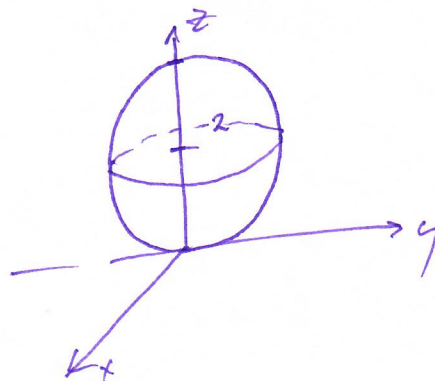
$$\text{or } \tan \phi = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \boxed{\phi = \pi/6}$$

Half-cone

c) Description of Sphere:

$$\boxed{x^2 + y^2 + z^2 = 4z} \Rightarrow \text{or } \boxed{x^2 + y^2 + (z-2)^2 = 4}$$

$$\rho = 4 \cos \phi \Rightarrow \boxed{\rho = 4 \cos \phi}$$



Integrals over regions E in Spherical coords.

(A) Spherical wedges:

$$E = \{ (r, \theta, \phi) \mid 0 \leq r \leq a, \alpha \leq \theta \leq \beta, c \leq \phi \leq d \}$$

$$\begin{aligned} \iiint_E f(x, y, z) dv &= \\ &= \int_c^d \int_\alpha^\beta \int_0^a f(r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi) r^2 \sin \phi dr d\theta d\phi \end{aligned}$$

(B) More general regions:

$$E = \{ (r, \theta, \phi) \mid \alpha \leq \theta \leq \beta, c \leq \phi \leq d, g_1(\theta, \phi) \leq r \leq g_2(\theta, \phi) \}$$

$$\begin{aligned} \iiint_E f(x, y, z) dv &= \\ &= \int_\alpha^\beta \int_c^d \int_{g_1(\theta, \phi)}^{g_2(\theta, \phi)} f(r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi) r^2 \sin \phi dr d\phi d\theta \end{aligned}$$

Spherical wedge

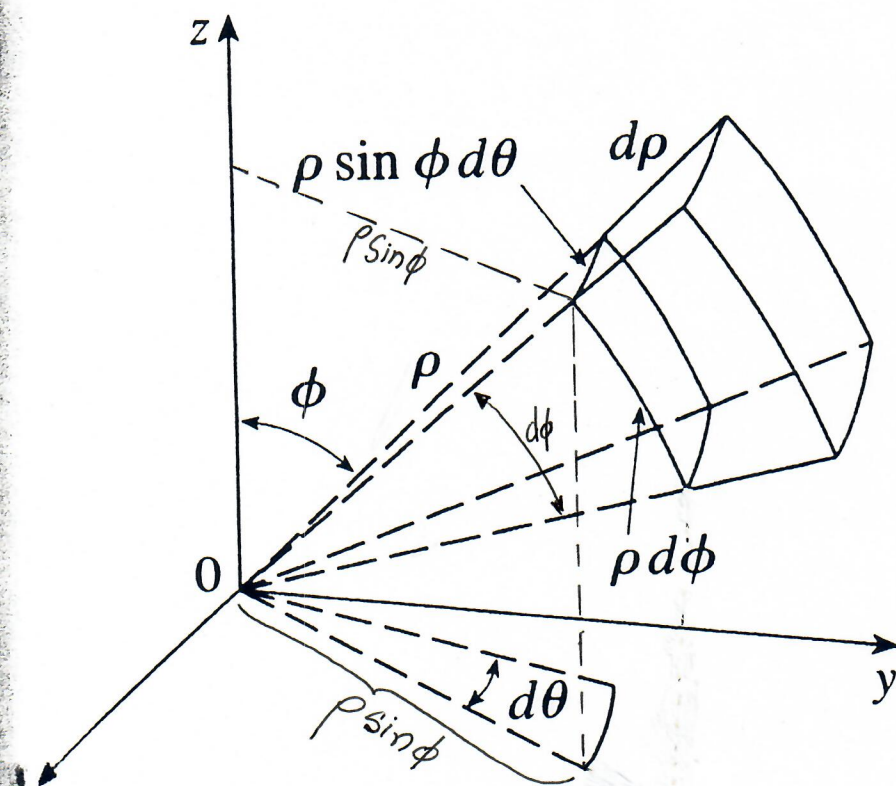


FIGURE 8

Volume element in spherical

coordinates: $dV = \rho^2 \sin \phi d\rho d\theta d\phi$

$$\Delta V_k \approx (\rho_k \Delta \phi_k) (\rho_k \sin(\phi_k) \Delta \theta_k) \Delta \rho_k = \rho_k^2 \sin \phi_k \Delta \rho_k \Delta \theta_k \Delta \phi_k$$

Using MVT, it can be shown that there is a point

$(\rho_k^*, \theta_k^*, \phi_k^*) \in V_k$ Satisfying

$$\Delta V_k = \rho_k^{*2} \sin(\phi_k^*) \Delta \rho_k \Delta \theta_k \Delta \phi_k. \text{ exactly}$$

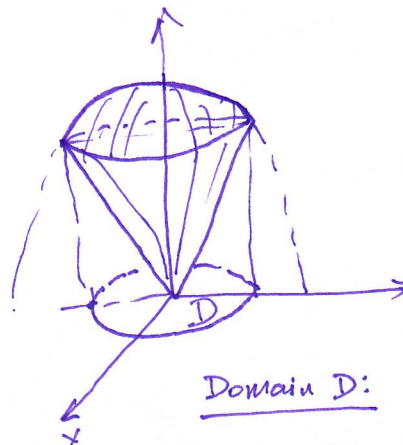
15.9 Spherical coords

3

(A) Ice-cream Cone

Vol. inside:

$$\left. \begin{array}{l} \text{Cone: } z = \sqrt{3x^2 + 3y^2} \\ \text{Sphere: } x^2 + y^2 + z^2 = 1 \end{array} \right\}$$

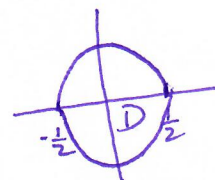


Domain D:

Combining $\rightarrow x^2 + y^2 + 3x^2 + 3y^2 = 1$

$$D: x^2 + y^2 = \frac{1}{4}$$

Circle of radius $\frac{1}{2}$.



I) In Cartesian:

$$\int_{-1/2}^{1/2} \int_{-\sqrt{1/4-x^2}}^{\sqrt{1/4-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{1-x^2-y^2}} dz dy dx$$

$$= \int_{-1/2}^{1/2} \int_{-\sqrt{1/4-x^2}}^{\sqrt{1/4-x^2}} \left(\sqrt{1-x^2-y^2} - \sqrt{3x^2+3y^2} \right) dy dx = ?$$

Hard to integrate!

II) In Spherical coords

ρ, θ, ϕ

$$E \equiv \left\{ (\rho, \theta, \phi) \mid 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/6 \right\}$$

$$\text{Vol.} = \iiint_E dv = \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^2 \sin \phi d\phi d\theta d\rho = \dots = \frac{\pi}{3} (2 - \sqrt{3}).$$

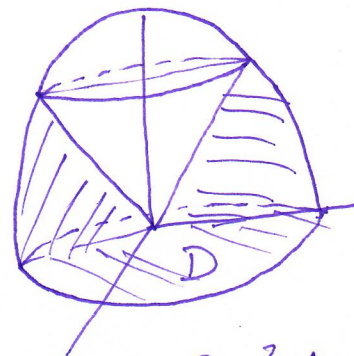
Easy algebra

In fact,
$$\text{Vol} = \int_0^1 \int_0^{2\pi} -\rho^2 \cos \phi \Big|_0^{\pi/6} d\theta d\rho = - \int_0^1 \int_0^{2\pi} \rho^2 \left(\frac{\sqrt{3}}{2} - 1 \right) d\theta d\rho$$

$$= - \int_0^1 2\pi \rho^2 \left(\frac{\sqrt{3}-2}{2} \right) d\rho = + \frac{\rho^3}{3} \Big|_0^1 (2-\sqrt{3}) \pi = \left(\frac{2-\sqrt{3}}{3} \right) \pi$$

(B) Vol. below Ice-Cream cone and inside
half-Sphere $z^2 = 3x^2 + 3y^2, x^2 + y^2 + z^2 = 1$

$$\text{Vol} = \int_0^1 \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho$$



$$D: x^2 + y^2 = 1$$

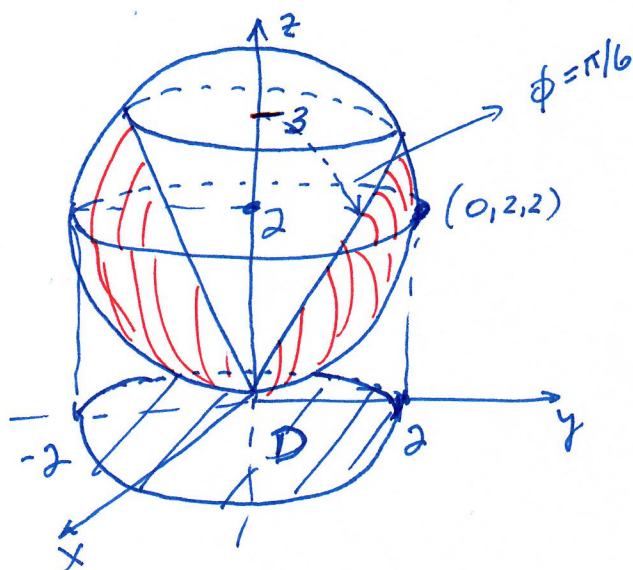
$$= \int_0^1 \int_0^{2\pi} \left. -\rho^2 \cos \phi \right|_{\pi/6}^{\pi/2} d\theta \, d\rho =$$

$$= - \int_0^1 \int_0^{2\pi} \rho^2 \left(-\frac{\sqrt{3}}{2} \right) d\theta \, d\rho = \int_0^1 2\pi \frac{\sqrt{3}}{2} \rho^2 \, d\rho = \pi \sqrt{3} \left. \frac{\rho^3}{3} \right|_0^1 =$$

$$= \boxed{\frac{\pi \sqrt{3}}{3}}$$

15.8 Spherical Coords. Triple Integrals.

- (C) Vol. of solid below cone: $z = \sqrt{3x^2 + 3y^2}$
and inside sphere: $x^2 + y^2 + z^2 = 4z$



$$x^2 + y^2 + z^2 = 4z \rightarrow x^2 + y^2 + (z-2)^2 = 4,$$

↓ Spherical

$$\rho = 4 \cos \phi,$$

$$0 \leq \phi \leq \pi/2$$

$$z = \sqrt{3(x^2 + y^2)}$$

↓ Spherical

$$\phi = \pi/6$$

Intersection of Sphere with Cone:

$$\text{Cone: } 3(x^2 + y^2) = z^2 \Rightarrow x^2 + y^2 = z^2/3$$

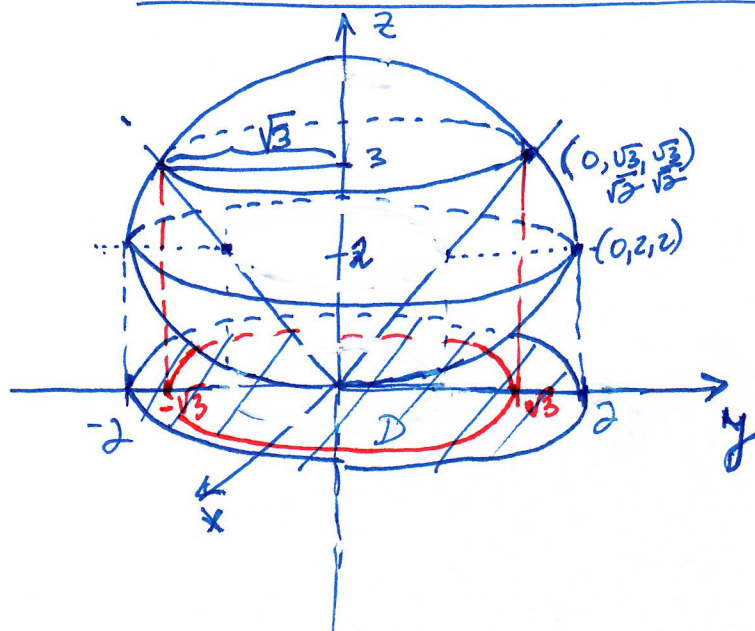
$$\text{Sphere: } (z-2)^2 = 4 - (x^2 + y^2) = 4 - z^2/3$$

$$\Rightarrow z^2 - 4z + 4 = 4 - z^2/3 \Rightarrow \frac{4}{3}z^2 - 4z = 0 \Leftrightarrow 4z\left(\frac{z}{3} - 1\right) = 0$$

$z=0, \boxed{z=3}$

$$\Rightarrow \boxed{\begin{matrix} x^2 + y^2 = 3 \\ z = 3 \end{matrix}}$$

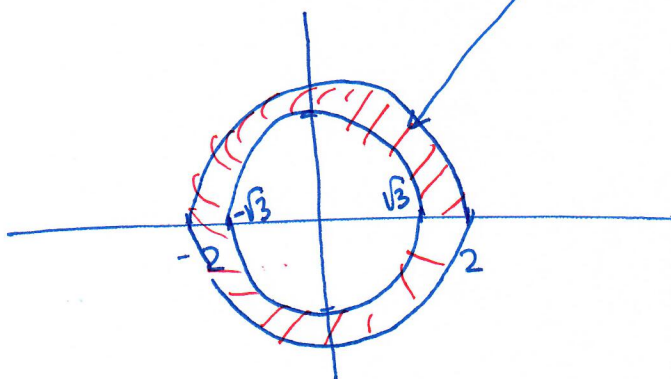
I) Therefore, integration in Cartesian coords:



Then,

$$I = \iint_{D_{\sqrt{3}}} \int_{-\sqrt{4-x^2-y^2}+2}^{\sqrt{3x^2+3y^2}} dz \, dA + \iint_{D_2 - D_{\sqrt{3}}} \int_{-\sqrt{4-x^2-y^2}+2}^{\sqrt{4-x^2-y^2}+2} dz \, dA$$

negative half sphere annular region



II) In Spherical Coordinates:

Cone: $\begin{cases} \phi = \pi/6 & \text{upper half cone.} \\ 0 \leq \theta \leq 2\pi \\ \rho \geq 0. \end{cases}$

b) Sphere: $x^2 + y^2 + z^2 = 4z$

$$\rho^2 = 4\rho \cos \phi \Rightarrow \boxed{\rho = 4 \cos \phi}, \quad 0 \leq \phi \leq \pi/2.$$

then,

$$I = \iiint_E dv = \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{4 \cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \left. \frac{\rho^3}{3} \right|_0^{4 \cos \phi} \sin \phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \frac{64}{3} \cos^3 \phi \sin \phi \, d\phi \, d\theta =$$

$$= - \int_0^{2\pi} \frac{64}{3} \frac{\cos^4 \phi}{4} \Big|_{\pi/6}^{\pi/2} d\theta = - \int_0^{2\pi} \frac{16}{3} \left(-\left(\frac{\sqrt{3}}{2}\right)^4 \right) d\theta =$$

$$= \int_0^{2\pi} \left(\frac{16}{3} \cdot \frac{9}{16} \right) d\theta = \frac{9}{3} (2\pi) = \boxed{\frac{18\pi}{3}}$$